THE UNCERTAINTY PRINCIPLE IN CLIFFORD ANALYSIS

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Abstract. In this paper, we provide the Heisenberg's inequality and the Hardy's theorem for the Clifford-Fourier transform on $\mathbb{R}^m$.

keywords: Clifford algebra, Clifford-Fourier transform, Heisenberg's inequality, Hardy's theorem.

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1. Introduction

In harmonic analysis, the uncertainty principle states that a non zero function and it's Fourier transform cannot both be very rapidly decreasing. This fact is expressed by two formulations of the uncertainty principle for the Fourier transform, Heisenberg's inequality and Hardy's theorem.
The classical Fourier transform is defined on $L^1(\mathbb{R})$ by:

$$\mathcal{F}(f)(x) = (2\pi)^{\frac{-1}{2}} \int_{\mathbb{R}} f(y) e^{-ixy} dy.$$

The Heisenberg's inequality asserts that for $f \in L^2(\mathbb{R})$

$$\int_{\mathbb{R}} ||x||^2 ||f(x)||^2 dx \int_{\mathbb{R}} ||\lambda||^2 ||\mathcal{F}(f)(\lambda)||^2 d\lambda \geq \frac{1}{4} \left[\int_{\mathbb{R}} ||f(x)||^2 dx \right]^2$$

with equality only if f almost everywhere equal to ce^{-px^2} for some $p > 0$. The proof of this inequality is given by Weyl in [9].
Hardy's theorem [5] says : If we suppose p and q be positive constants and f a function on the real line satisfying $|f(x)| \leq Ce^{-px^2}$ and $|\mathcal{F}(f)(\lambda)| \leq Ce^{-q\lambda^2}$ for some positive constant C, then (i) $f = 0$ if $pq > \frac{1}{4}$; (ii) $f = Ae^{-px^2}$ for some constant A if $pq = \frac{1}{4}$; (iii) there are many f if $pq < \frac{1}{4}$.

In Dunkl theory, the Heisenberg's inequality for the Dunkl transform was proved by Rösler in [6] and Shimeno in [7]. Hardy's theorem for the Dunkl transform was given by Shimeno in [7].

In Clifford analysis, the Clifford-Fourier transform was introduced and studied by De Bie and Xu in [3]. In [4], differents properties of the Clifford-Fourier transform and the Plancherel's theorem were established.

In this paper, we provide an analogues of the Heisenberg's inequality and Hardy's theorem for the Clifford-Fourier transform on $\mathbb{R}^m$.
Our paper is orgonized as follows. In section 2, we review basic notions and notations related to the Clifford algebra. In section 3, we recall some results and properties for the Clifford-Fourier transform useful in the sequel. Also, we provide some new properties associated to the kernel of the integral Clifford-Fourier transform and the Clifford-Fourier transform. In section 4, we prove the Heisenberg's inequality for the Clifford-Fourier transform. In section 5, we provide the Hardy's theorem in Clifford analysis on $\mathbb{R}^m$, with m even.

2. Notations and preliminaries

We introduce the Clifford algebra $Cl_{0,m}$ over $\mathbb{R}^m$ as a non commutative algebra generated by the basis $\{e_1, .., e_m\}$ satisfying the rules:

$$\left\{ \begin{array}{ll} e_i e_j = -e_j e_i, & \text{if} \quad i \neq j; \\ e_i^2 = -1, & \forall 1 \leq i \leq m. \end{array} \right. \tag{2.1}$$

This algebra can be decomposed as

$$Cl_{0,m} = \oplus_{k=0}^{m} Cl_{0,m}^k, \tag{2.2}$$

with $Cl_{0,m}^k$ the space of vectors defined by

$$Cl_{0,m}^k = span\{e_{i_1}..e_{i_k}, i_1 < .. < i_k\}. \tag{2.3}$$

Hence, $\{1, e_1, e_2, .., e_{12}, .., e_{12..m}\}$ constitute a basis of $Cl_{0,m}$.
A Clifford number x in $Cl_{0,m}$ is written as follows :

$$x = \sum_{A \in J} x_A e_A, \tag{2.4}$$

where $J := \{0, 1, .., m, 12, .., 12..m\}$, x_A real number and e_A belong to the basis of $Cl_{0,m}$ defined above.
The norm of such element x is given by :

$$||x||_c = \left(\sum_{A \in J} x_A^2 \right)^{\frac{1}{2}}. \tag{2.5}$$

In particular, if x is a vector in $Cl_{0,m}$ then

$$||x||_c^2 = -x^2.$$

The Dirac operator is defined by :

$$\partial_x = \sum_{i=1}^{m} e_i \partial_{x_i}. \tag{2.6}$$

We introduce respectively the Gamma operator associated to a vector x, the inner product and the wedge product of two vectors x and y :

$$\Gamma_x := -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}); \tag{2.7}$$

$$< x, y >:= \sum_{j=1}^{m} x_j y_j = \frac{-1}{2}(xy + yx); \tag{2.8}$$

$$x \wedge y := \sum_{j<k} e_j e_k (x_j y_k - x_k y_j) = \frac{1}{2}(xy - yx). \tag{2.9}$$

Every function f defined on $\mathbb{R}^m$ and taking values in $Cl_{0,m}$ can be decomposed as :

$$f(x) = f_0(x) + \sum_{i=1}^{m} e_i f_i(x) + \sum_{i<j} e_i e_j f_{ij}(x) + .. + e_1..e_m f_{1..m}(x), \tag{2.10}$$

with $f_0, f_i, .., f_{1..m}$ all real-valued functions.
We denote by :
$\bullet \mathcal{P}_k$ the space of homogenious polynomials of degree k taking values in $Cl_{0,m}$,
$\bullet \mathcal{M}_k := Ker\partial_x \cap \mathcal{P}_k$ the space of spherical monogenics of degree k,
$\bullet B(\mathbb{R}^m) \otimes Cl_{0,m}$ a class of integrable functions taking values in $Cl_{0,m}$ and satisfying

$$\int_{\mathbb{R}^m} (1 + ||y||_c)^{\frac{m-2}{2}} ||f(y)||_c dy < \infty, \tag{2.11}$$

$\bullet L_2(\mathbb{R}^m) \otimes Cl_{0,m}$ the space of square integrable functions taking values in $Cl_{0,m}$ with the norm equiped

$$||f||_{2,c} = \left(\int_{\mathbb{R}^m} ||f(x)||_c^2 dx \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^m} \left(\sum_{A \in J} (f_A(x))^2 \right)^{\frac{1}{2}} dx \right)^{\frac{1}{2}}, \tag{2.12}$$

where $J = \{0, 1, .., m, 12, 13, 23.., 12..m\}$,
$\bullet S(\mathbb{R}^m)$ the Schwartz space of infinitely differentiable functions on $\mathbb{R}^m$ which are rapidly decreasing as their derivatives.

3. Clifford-Fourier Transform

Definition 3.1. *Let $f \in B(\mathbb{R}^m) \otimes Cl_{0,m}$. We define the Clifford-Fourier transform $\mathcal{F}_\pm$ of f by (see* [3]*) :*

$$\mathcal{F}_\pm(f)(y) = (2\pi)^{\frac{-m}{2}} \int_{\mathbb{R}^m} K_\pm(x, y) f(x) dx, \tag{3.1}$$

where

$$K_\pm(x, y) = e^{\mp i \frac{\pi}{2} \Gamma_y} e^{-i<x,y>}. \tag{3.2}$$

Lemma 3.1. *Assume m be even and upper than two. Then*

$$||K_-(x,y)||_c \le Ce^{||x||_c||y||_c}, \qquad \forall x,y \in \mathbb{R}^m, \tag{3.3}$$

with C a positive constant.

Proof. Recall that the kernel $K_-(x,y)$ can be decomposed as

$$K_-(x,y) = K_0^-(x,y) + \sum_{i<j} e_i e_j K_{ij}^-(x,y),$$

with $K_0^-(x,y)$ and $K_{ij}^-(x,y)$ scalar functions. Moreover, by theorem 5.3 in [3] we have for m even and $x,y \in \mathbb{R}^m$:

$$||K_0^-(x,y)||_c \le c(1+\|x\|_c)^{\frac{m-2}{2}}(1+\|y\|_c)^{\frac{m-2}{2}},$$

$$||K_{ij}^-(x,y)||_c \le c(1+\|x\|_c)^{\frac{m-2}{2}}(1+\|y\|_c)^{\frac{m-2}{2}}, i \neq j,$$

which complete the proof.

∎

Lemma 3.2. *Let $c > 0$, then*

$$K_\pm(c^{-1}x, cy) = K_\pm(x,y), \quad forall x,y \in \mathbb{R}^m. \tag{3.4}$$

Proof. By proposition 3.4 in [3], it is enough to prove the lemma just for K_-.
As the kernel of the Clifford-Fourier transform is given by (see theorem 3.2 in [3])

$$K_-(x,y) = A_\lambda(w,z) + B_\lambda(w,z) + x \wedge y C_\lambda(w,z)$$

where

$$A_\lambda(w,z) = 2^{\lambda-1}\Gamma(\lambda+1)\sum_{k=0}^{\infty}(i^{2\lambda+2} + (-1)^k)z^{-\lambda}J_{k+\lambda}(z)C_k^\lambda(w),$$

$$B_\lambda(w,z) = -2^{\lambda-1}\Gamma(\lambda)\sum_{k=0}^{\infty}(k+\lambda)(i^{2\lambda+2} - (-1)^k)z^{-\lambda}J_{k+\lambda}(z)C_k^\lambda(w),$$

$$C_\lambda(w,z) = -2^{\lambda-1}\Gamma(\lambda)\sum_{k=0}^{\infty}(i^{2\lambda+2} + (-1)^k)z^{-\lambda-1}J_{k+\lambda}(z)\left(\frac{d}{dw}C_k^\lambda\right)(w),$$

with $z = ||x||_c||y||_c$, $zw =< x,y >$ and $\lambda = \frac{m-2}{2}$. Thus, it suffies to substitute x and y by $c^{-1}x$ and cy. ∎

Lemma 3.3. *Let c be positive constant and $f \in B(\mathbb{R}^m) \otimes Cl_{0,m}$.*
Assume $f_c(x) := f(cx)$ for all $x \in \mathbb{R}^m$. Then

$$\mathcal{F}_\pm(f_c)(\lambda) = c^m \mathcal{F}_\pm(f)(c^{-1}\lambda). \tag{3.5}$$

Proof. Applying a variable change to definition 3.1, we get

$$\mathcal{F}_{\pm}(f_c)(\lambda) = (2\pi)^{\frac{-m}{2}} c^m \int_{\mathbb{R}^m} K_{\pm}(\frac{x}{c}, \lambda) f(x) dx.$$

By lemma 3.2, we find

$$\mathcal{F}_{\pm}(f_c)(\lambda) = (2\pi)^{\frac{-m}{2}} c^m \int_{\mathbb{R}^m} K_{\pm}(x, c^{-1}\lambda) f(x) dx.$$

We conclude. ∎

Theorem 3.4 (The case $i = \frac{m-2}{2}$, theorem 6.4 in [4])**.** *The Clifford-Fourier transform $\mathcal{F}_{\pm}$ can be extended from $\mathcal{S}(\mathbb{R}^m) \otimes Cl_{0,m}$ to a continuous map on $L_2(\mathbb{R}^m) \otimes Cl_{0,m}$. In particular, when m is even, for all $f \in L_2(\mathbb{R}^m) \otimes Cl_{0,m}$, we have :*

$$||\mathcal{F}_{\pm}(f)||_{2,c} = ||f||_{2,c}. \tag{3.6}$$

Theorem 3.5. *The Clifford-Fourier transform coincides with the classical Fourier transform for radial functions (see*[4]*).*
In particular,

$$\mathcal{F}_{-}\left(e^{-\frac{||.||^2}{2}}\right)(x) = e^{-\frac{||x||^2}{2}}, \quad x \in \mathbb{R}^m. \tag{3.7}$$

4. Heisenberg's inequalities

In this section, we establish the Heisenberg's inequality for the Clifford-Fourier transform. The proof is based on the consideration of a basis of $L_2(\mathbb{R}^m) \otimes Cl_{0,m}$ which is expressed in terms of Laguerre polynomials and recurrence relation's. Indeed it is an analogue of the proof of Heisenberg's inequality in the Dunkl analysis (see [7]).
We introduce a basis $\{\psi_{j,k,l}\}$ of $\mathcal{S}(\mathbb{R}^m) \otimes Cl_{0,m}$ which is defined as follows :

$$\psi_{2j,k,l}(x) := L_j^{\frac{m}{2}+k-1}(||x||_c^2) M_k^{(l)} e^{-\frac{||x||_c^2}{2}}, \tag{4.1}$$

$$\psi_{2j+1,k,l}(x) := L_j^{\frac{m}{2}+k}(||x||_c^2) x M_k^{(l)} e^{-\frac{||x||_c^2}{2}}. \tag{4.2}$$

where $j, k \in \mathbb{N}$, $\{M_k^{(l)} \in \mathcal{M}_k; \quad l = 1, .., dim\mathcal{M}_k\}$ is an orthogonal basis for $\mathcal{M}_k$ and L_j^{α} the classical Laguerre polynomials.

Lemma 4.1. $\{\psi_{j,k,l}\}$ *is an orthogonal basis of* $L_2(\mathbb{R}^m) \otimes Cl_{0,m}$.

Proof. Sommen and Brackx in [1] give an orthogonal basis of $L_2(\mathbb{R}^m) \otimes Cl_{0,m}$ in terms of generalized Clifford polynomials as follows:

$$\psi_{j,k,l}(x) = H_{j,k,l}(\sqrt{2}x) M_k^{(l)}(\sqrt{2}x) e^{-\frac{||x||_c^2}{2}},$$

with $H_{j,k,l}$ the Hermite polynomials and $M_k^{(l)} \in \mathcal{M}_k$; $l = 1, .., dim(\mathcal{M}_k)$; $j, k \in \mathbb{N}$.
Using the relations of the Hermite polynomials defined in [2] ,we get the desired result. ∎

Theorem 4.2. *(see* [3]*).* For the basis $\{\psi_{j,k,l}\}$ of $\mathcal{S}(\mathbb{R}^m)\otimes Cl_{0,m}$, one has

$$\mathcal{F}_{\pm}(\psi_{2j,k,l}) = (-1)^{j+k}(\mp 1)^k \psi_{2j,k,l}, \tag{4.3}$$

$$\mathcal{F}_{\pm}(\psi_{2j+1,k,l}) = i^m(-1)^{j+1}(\mp 1)^{k+m-1}\psi_{2j+1,k,l}. \tag{4.4}$$

Lemma 4.3. *Let a, b, c are positive real numbers such that*

$$(\frac{a}{t})^2 + b^2t^2 \geq c, \forall t > 0,$$

then

$$ab \geq \frac{c}{2}.$$

Proof. Let

$$h : \mathbb{R}_+^* \to \mathbb{R}$$
$$t \mapsto (\frac{a}{t})^2 + b^2t^2.$$

It is clear that h is a differentiable function and the corresponding derivative is given by :

$$\begin{aligned} h'(t) &= 2b^2t - 2a^2t^{-3} \\ &= 2t^{-3}(b^2t^4 - a^2) \\ &= 2t^{-3}(bt^2 - a)(bt^2 + a). \end{aligned}$$

Since $h(t) \geq c, \forall t > 0$, then

$$2ab = h(\sqrt{\frac{a}{b}}) = \inf_{t>0} h(t) \geq c.$$

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Theorem 4.4. *Let $f \in L_2(\mathbb{R}^m)\otimes Cl_{0,m}$. Then*

$$\int_{\mathbb{R}^m} ||x||_c^2 ||f(x)||_c^2 dx \int_{\mathbb{R}^m} ||\lambda||_c^2 ||\mathcal{F}(f)(\lambda)||_c^2 d\lambda \geq \frac{m^2}{4}\left[\int_{\mathbb{R}^m} ||f(x)||_c^2 dx\right]^2, \tag{4.5}$$

with equality if $f = ce^{-p||x||_c^2}$, almost everywhere for some $p > 0$.

Proof. We start by proving the inequality .
The application of the recurrence relation for the Laguerre polynomials

$$tL_n^{(\alpha)}(t) = -(n+1)L_{n+1}^{(\alpha)}(t) + (\alpha+2n+1)L_n^{(\alpha)}(t) - (\alpha+n)L_{n-1}^{(\alpha)}(t)$$

to the basis $\{\psi_{j,k,l}\}$ and relations (4.1) and (4.2) leads to :

$$||x||_c^2\psi_{2j,k,l}(x) = -(j+1)\psi_{2(j+1),k,l} + (\frac{m}{2}+k+2j)\psi_{2j,k,l} - (\frac{m}{2}+k-1+j)\psi_{2(j-1),k,l} \tag{4.6}$$

(4.7)

$$||x||_c^2\psi_{2j+1,k,l}(x) = -(j+1)\psi_{2j+3,k,l} + (\frac{m}{2}+k+2j+1)\psi_{2j+1,k,l} - (\frac{m}{2}+k+j)\psi_{2j-1,k,l}.$$

We set $\psi_{-1,k,l}=0$ and $\psi_{-2,k,l}=0$.

As $\{\psi_{j,k,l}\}$ is a basis of $L_2(\mathbb{R}^m)\otimes Cl_{0,m}$, every function f in $L_2(\mathbb{R}^m)\otimes Cl_{0,m}$ can be writen as follows :

$$f=\sum_{j,k,l}a_{j,k,l}\psi_{j,k,l}=\sum_{2j,k,l}a_{2j,k,l}\psi_{2j,k,l}+\sum_{2j+1,k,l}a_{2j+1,k,l}\psi_{2j+1,k,l}, \tag{4.8}$$

where $a_{j,k,l}=<f,\psi_{j,k,l}>$.
Theorem 4.2 implies that :
(4.9)

$$\mathcal{F}_{\pm}(f)=\sum_{j,k,l}a_{2j,k,l}(-1)^{j+k}(\mp 1)^{k}\psi_{2j,k,l}+\sum_{j,k,l}a_{2j+1,k,l}i^{m}(-1)^{j+1}(\mp 1)^{k+m-1}\psi_{2j+1,k,l}.$$

Using the last relations (4.6) and (4.7) and the ortogonality of the basis $\{\psi_{j,k,l}\}$ in $L_2(\mathbb{R}^m)\otimes Cl_{0,m}$, we get :

$$\begin{aligned}|||.|_c f||^2_{2,c}&=\sum_{j,k,l}a_{j,k,l}<||.||^2_c\psi_{j,k,l},f>\\&=\sum_{j,k,l}a_{2j,k,l}<||.||^2_c\psi_{2j,k,l},f>+\sum_{j,k,l}a_{2j+1,k,l}<||.||^2_c\psi_{2j+1,k,l},f>.\end{aligned}$$

Thus :

$$\begin{aligned}|||.|f||^2_{2,c}=&-\sum_{j,k,l}(j+1)a_{2j,k,l}a_{2j+2,k,l}||\psi_{2j+2,k,l}||^2_c+\sum_{j,k,l}a^2_{2j,k,l}(\frac{m}{2}+k+2j)||\psi_{2j,k,l}||^2_c\\&-\sum_{j,k,l}(\frac{m}{2}+k+j-1)a_{2j,k,l}a_{2j-2,k,l}||\psi_{2j-2,k,l}||^2_c-\sum_{j,k,l}(j+1)a_{2j+1,k,l}a_{2j+3,k,l}||\psi_{2j+3,k,l}||^2_c\\&+\sum_{j,k,l}a^2_{2j+1,k,l}(\frac{m}{2}+k+2j+1)||\psi_{2j+1,k,l}||^2_c-\sum_{j,k,l}a_{2j+1,k,l}a_{2j-1,k,l}(\frac{m}{2}+k+j)||\psi_{2j-1,k,l}||^2_c.\end{aligned}$$

With the same manner, we find :

$$\begin{aligned}|||.|^2_c\mathcal{F}_{\pm}(f)||^2_{2,c}=&\sum_{j,k,l}(j+1)a_{2j,k,l}a_{2j+2,k,l}||\psi_{2j+2,k,l}||^2_c+\sum_{j,k,l}a^2_{2j,k,l}(\frac{m}{2}+k+2j)||\psi_{2j,k,l}||^2_c\\&+\sum_{j,k,l}(\frac{m}{2}+k+j-1)a_{2j,k,l}a_{2j-2,k,l}||\psi_{2j-2,k,l}||^2_c+\sum_{j,k,l}(j+1)a_{2j+1,k,l}a_{2j+3,k,l}||\psi_{2j+3,k,l}||^2_c\\&+\sum_{j,k,l}a^2_{2j+1,k,l}(\frac{m}{2}+k+2j+1)||\psi_{2j+1,k,l}||^2_c+\sum_{j,k,l}a_{2j+1,k,l}a_{2j-1,k,l}(\frac{m}{2}+k+j)||\psi_{2j-1,k,l}||^2_c.\end{aligned}$$

Now, by collecting the two equalities we get :

$$\begin{aligned}|||.|_c f||^2_{2,c}+|||.|^2_c\mathcal{F}_{\pm}f||^2_{2,c}=2(&\sum_{j,k,l}a^2_{2j,k,l}(\frac{m}{2}+k+2j)||\psi_{2j,k,l}||^2_c\\&+\sum_{j,k,l}a^2_{2j+1,k,l}(\frac{m}{2}+k+2j+1)||\psi_{2j+1,k,l}||^2_c)\end{aligned} \tag{4.10}$$

$$\geq 2\sum_{j,k,l} a_{j,k,l}^2(\frac{m}{2}+k+2j)||\psi_{j,k,l}||_c^2$$
$$\geq m||f||_{2,c}^2.$$

Let $f \in L_2(\mathbb{R}^m)\otimes Cl_{0,m}$ and $k>0$. We define the function f_k by :

$$f_k(x) := f(kx). \tag{4.11}$$

Then, we have :

$$|||.|f_k||_{2,c}^2 = \int_{\mathbb{R}^m} ||x||_c^2||f_k(x)||_c^2dx = \int_{\mathbb{R}^m} ||x||_c^2||f(kx)||_c^2dx \tag{4.12}$$
$$= k^{m-2}\int_{\mathbb{R}^m} ||x||_c^2||f(x)||_c^2dx = k^{m-2}|||.|f||_{2,c}^2.$$

By lemma 3.3, we find :

$$|||.|\mathcal{F}_\pm(f_k)||_{2,c}^2 = \int_{\mathbb{R}^m} ||\lambda||_c^2||\mathcal{F}_\pm(f_k)(\lambda)||_c^2d\lambda \tag{4.13}$$
$$= k^{2m}\int_{\mathbb{R}^m} ||\lambda||_c^2||\mathcal{F}_\pm(f)(k^{-1}\lambda)||_c^2d\lambda$$
$$= k^{2+m}\int_{\mathbb{R}^m} ||\lambda||_c^2||\mathcal{F}_\pm(f)(\lambda)||_c^2d\lambda$$
$$= k^{m+2}|||.|\mathcal{F}_\pm(f)||_{2,c}^2.$$

Using relation (4.10), we obtain :

$$k^{-2}|||.|f||_{2,c}^2 + k^2|||.|\mathcal{F}_\pm(f|)|_{2,c}^2 \geq m||f||_{2,c}^2. \tag{4.14}$$

Lemma 4.3, completes the proof of inequality.

Now, we treat the case of equality for a function f almost everywhere equal to $ce^{-p||x||_c^2}$ for some positive constant p. We check the case $p=\frac{1}{2}$. Then, we generalize it for some positive p.

Let $f_1 = ce^{\frac{-||x||_c^2}{2}}$, then

$$|||.|f_1||_{2,c}^2 = \int_{\mathbb{R}^m} ||x||_c^2c^2e^{-||x||_c^2}dx.$$

By Theorem 3.5, we have :

$$|||.|\mathcal{F}_\pm(f_1)||_{2,c}^2 = |||.|f_1||_{2,c}^2.$$

Thus

$$|||.|f_1||_{2,c}^2|||.|\mathcal{F}_\pm f_1||_{2,c}^2 = \left(\int_{\mathbb{R}^m} ||x||_c^2c^2e^{-||x||_c^2}dx\right)^2.$$

Since $\partial_x(x) = -m$, $x^2 = -||x||_c^2$ and $\partial_x(\frac{1}{2}e^{-||x||_c^2}) = xe^{-||x||_c^2}$, an integration by parts shows that :

$$\int_{\mathbb{R}^m} ||x||_c^2e^{-||x||_c^2}dx = -\frac{m}{2}\int_{\mathbb{R}^m} e^{-||x||_c^2}dx.$$

Hence

$$|||.|f_1||^2_{2,c}|||.|\mathcal{F}_\pm(f_1)||^2_{2,c} = \frac{m^2}{4}||f_1||^4_{2,c}. \tag{4.15}$$

For the general case, we just find relations between the norm $|||.|f||^2_{2,c}$ and $|||.|f_1||^2_{2,c}$, also between $|||.|\mathcal{F}_\pm(f)||^2_{2,c}$ and $|||.|\mathcal{F}_\pm(f_1)||^2_{2,c}$.

Put $f(x) = ce^{-p||x||^2_c}$. We recall that the Clifford-Fourier transform for such f is defined by

$$\mathcal{F}_\pm(f)(x) = (2\pi)^{\frac{-m}{2}} \int_{\mathbb{R}^m} K_\pm(y,x)ce^{-p||y||^2_c}dy.$$

Since $f(x) = f_1(\sqrt{2p}x)$, for all $x \in \mathbb{R}^m$, lemma 3.3 leads to

$$\mathcal{F}_\pm(f)(x) = (\sqrt{2p})^{-m}\mathcal{F}_\pm(f_1)(\frac{x}{\sqrt{2p}}).$$

Consequently

$$\begin{aligned}|||.|\mathcal{F}_\pm(f)||^2_{2,c} &= \int_{\mathbb{R}^m} ||x||^2_c||\mathcal{F}_\pm f(x)||^2_c dx \\ &= (2p)^{-m}\int_{\mathbb{R}^m} ||x||^2_c||\mathcal{F}_\pm f_1(\frac{x}{\sqrt{2p}})||^2_c dx.\end{aligned}$$

By a change of variable, we have :

$$|||.|\mathcal{F}_\pm(f)||^2_{2,c} = (\sqrt{2p})^{-m+2}|||.|\mathcal{F}_\pm(f_1)||^2_{2,c}.$$

By another change of variable, we obtain

$$|||.|f||^2_{2,c} = (\sqrt{2p})^{-m-2}|||.|f_1||^2_{2,c}.$$

The required equality is deduced by relation (4.15).

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5. Hardy's theorem

In this section, we prove Hardy's theorem for the Clifford-Fourier transform.

Theorem 5.1. *Let p and q are positive constants and m is an even integer upper than two. Assume $f \in B(\mathbb{R}^m) \otimes Cl_{0,m}$ such that :*

$$||f(x)||_c \leq Ce^{-p||x||^2_c}, \quad x \in \mathbb{R}^m \tag{5.1}$$

and

$$||\mathcal{F}_\pm(f)(\lambda)||_c \leq Ce^{-q||\lambda||^2_c}, \quad \lambda \in \mathbb{R}^m, \tag{5.2}$$

for some positive constant C. Then, three cases can occur :

i) If $pq > \frac{1}{4}$, then $f = 0$.
ii) If $pq = \frac{1}{4}$, then $f(x) = Ae^{-\frac{||x||^2_c}{2}}$, where A is a constant.
iii) If $pq < \frac{1}{4}$, then there are many f.

Proof. The functions $\{\psi_{j,k,l}\}$ defined by (4.1) and (4.2) give an infinite number of examples for iii).
It is well known that by scaling (3.5), we may assume $p = q$ without loss of generality.
The proof of i) is a simple deduction of ii).

Assume $p = q = \frac{1}{2}$. Let $f \in B(\mathbb{R}^m) \otimes Cl_{0,m}$ satisfying (5.1), we have for $\lambda \in \mathbb{R}^m \otimes \mathbb{C}$

$$\begin{aligned} ||\mathcal{F}_{\pm}(f)(\lambda)||_c &\leq (2\pi)^{\frac{-m}{2}} \int_{\mathbb{R}^m} ||K_{\pm}(x,\lambda)||_c ||f(x)||_c dx \\ &\leq C(2\pi)^{\frac{-m}{2}} \int_{\mathbb{R}^m} ||K_{\pm}(x,\lambda)||_c e^{-\frac{||x||_c^2}{2}} dx. \end{aligned}$$

Applying lemma 3.1, we get :

$$\begin{aligned} ||\mathcal{F}_{\pm}(f)(\lambda)||_c &\leq C(2\pi)^{\frac{-m}{2}} \int_{\mathbb{R}^m} e^{||x||_c ||\lambda||_c - \frac{||x||_c^2}{2}} dx \\ &\leq e^{\frac{||\lambda||_c^2}{2}} C(2\pi)^{\frac{-m}{2}} \int_{\mathbb{R}^m} e^{\frac{-(||x||_c - ||\lambda||_c)^2}{2}} dx. \end{aligned}$$

Thus

$$||\mathcal{F}_{\pm}(f)(\lambda)||_c \leq C' e^{\frac{||\lambda||_c^2}{2}}, \tag{5.3}$$

where C' is a positive constant.
As $\mathcal{F}_{\pm}(f)$ is an entire function satisfying relation (5.2) and (5.3), lemma 2.1 in[8] allows to express $\mathcal{F}_{\pm}(f)$ as follows :

$$\mathcal{F}_{\pm}(f)(x) = Ae^{-\frac{||x||_c^2}{2}},$$

with A is a constant.
Theorem 3.5 completes the proof.

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